Calculate the limit
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$\lim_{n\to\infty}\frac{1}{n}\sqrt[n]{\frac{(2n)!}{n!}}$.

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Solution1.

Let $a_n := \frac{(2n)!}{n^n n!}$. Since $\frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{(n+1)^{n+1}(n+1)!}\cdot\frac{n^n n!}{(2n)!} = \frac{2(2n+1)}{\left(1+\frac{1}{n}\right)^n(n+1)}$ and $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 2\lim_{n\to\infty}\frac{2n+1}{\left(1+\frac{1}{n}\right)^n(n+1)} = \frac{4}{e}$ then* by GM limit Theorem (Multiplicative Cesaro's Theorem) $\lim_{n\to\infty}\frac{1}{n}\left(\frac{(2n)!}{n!}\right)^{1/n} = \lim_{n\to\infty}\sqrt[n]{a_n} = \frac{4}{e}$.

*Geometric Mean limit Theorem: If $\lim_{n\to\infty} a_n = a$ then $\lim_{n\to\infty}\sqrt[n]{a_1 a_2 \ldots a_n} = a$.

In particular if $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = a$, then by replacing $(a_1, a_2, , \ldots, a_n)$ with $\left(a_1, \frac{a_2}{a_1}, \frac{a_3}{a_2}, \ldots, \frac{a_n}{a_{n-1}}\right)$ we obtain $\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{a_1\cdot\frac{a_2}{a_1}\cdot\frac{a_3}{a_2}\cdot\ldots\cdot\frac{a_n}{a_{n-1}}} = a$.

Solution 2.

Using double inequality $\left(\frac{n+1}{e}\right)^n < n! < \left(\frac{n+1}{e}\right)^n(n+1), \forall n \in \mathbb{N}$ (follows from the double inequality $\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}$) we obtain

$$\left(\frac{2n+1}{e}\right)^{2n}\cdot\frac{e^n}{(n+1)^{n+1}} < \frac{(2n)!}{n!} < \left(\frac{2n+1}{e}\right)^{2n}(2n+1)\cdot\left(\frac{e}{n+1}\right)^n \Leftrightarrow$$

$$\frac{(2n+1)^{2n}}{e^n(n+1)^{n+1}} < \frac{(2n)!}{n!} < \frac{(2n+1)^{2n+1}}{e^n(n+1)^n} \Leftrightarrow$$

$$\frac{(2n+1)^2}{en(n+1)\sqrt[n]{n+1}} < \frac{1}{n}\sqrt[n]{\frac{(2n)!}{n!}} < \frac{(2n+1)^2\sqrt[n]{2n+1}}{en(n+1)}.$$

Since $\lim_{n\to\infty}\sqrt[n]{n+1} = \lim_{n\to\infty}\sqrt[n]{2n+1} = 1,\ \lim_{n\to\infty}\frac{(2n+1)^2}{n(n+1)} = 4$ then by Squeeze Principle

$\lim_{n\to\infty}\frac{1}{n}\sqrt[n]{\frac{(2n)!}{n!}} = \frac{4}{e}$.